Proving that Fib(n) $= \frac{\phi^n - \gamma^n}{\sqrt{5}}$ where $\phi = \frac{1+\sqrt{5}}{2}$ and $\gamma = \frac{1-\sqrt{5}}{2}$.

I prove this using the inductive defination of Fib(n) and strong induction. Evaluating the base case is left to the reader. So, given Fib(a) $= \frac{\phi^a - \gamma^a}{\sqrt{5}}$ for all $a < n$ I wish to show Fib(n) as above.

From the defination Fib(n) = Fib(n-1) + Fib(n-2) $= \frac{\phi^{n-1} - \gamma^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2} - \gamma^{n-2}}{\sqrt{5}}$.

I will show that $\frac{\phi^{n-1} - \gamma^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2} - \gamma^{n-2}}{\sqrt{5}} = \frac{\phi^n - \gamma^n}{\sqrt{5}}$ if, and only if, True.

Multiply everything by $\sqrt{5}$: $\phi^{n-1} - \gamma^{n-1} + \phi^{n-2} - \gamma^{n-2} = \phi^n - \gamma^n$

And expand: $\frac{(1+\sqrt{5})^{n-1}}{2^{n-1}} - \frac{(1-\sqrt{5})^{n-1}}{2^{n-1}} + \frac{(1+\sqrt{5})^{n-2}}{2^{n-2}} - \frac{(1-\sqrt{5})^{n-2}}{2^{n-2}} = \frac{(1+\sqrt{5})^n}{2^n} - \frac{(1-\sqrt{5})^n}{2^n}$

Multiply top and bottom of each fraction to get 2^n on the bottom of everything and multiply through: $2(1+\sqrt{5})^{n-1} - 2(1-\sqrt{5})^{n-1} + 4(1+\sqrt{5})^{n-2} - 4(1-\sqrt{5})^{n-2} = (1+\sqrt{5})^n - (1-\sqrt{5})^n$.

Rearrange: $(1+\sqrt{5})^{n-2}(2(1+\sqrt{5}) + 4 - (1+\sqrt{5})^2) = (1-\sqrt{5})^{n-2}(2(1-\sqrt{5}) + 4 - (1-\sqrt{5})^2)$

Simplify: $(1+\sqrt{5})^{n-2}(0) = (1-\sqrt{5})^{n-2}(0)$

Simplify: $0 = 0$

Simplify: True. Finished

Fib(n) is the closest integer to $\phi^n/\sqrt{5}$. We know that Fib(n) $= \frac{\phi^n - \gamma^n}{\sqrt{5}} = \frac{\phi^n}{\sqrt{5}} - \frac{\gamma^n}{\sqrt{5}}$. Since $\frac{\gamma^n}{\sqrt{5}} < 0.5$ (for $n \geq 0$) this is true.